A PROOF THAT THOMPSON'S GROUPS HAVE INFINITELY MANY RELATIVE ENDS

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Abstract. We show that each of Thompson's groups F, T, and V has infinitely many ends relative to the groups $F_{[0,1/2]}$, $T_{[0,1/2]}$, and $V_{[0,1/2)}$ (respectively).

As an application, we simplify the proof, due to Napier and Ramachandran, that F, T, and V are not Kähler groups.

We go on to show that Thompson's groups T and V have Serre's property FA. The main theorems together answer a question on Bestvina's problem list that was originally posed by Mohan Ramachandran.

1. Introduction

Thompson's group F is the group of piecewise linear homeomorphisms h of the unit interval such that: i) each of the finitely many places at which h fails to be differentiable are dyadic rational numbers, and ii) at every other point $x \in [0,1]$, $h'(x) \in \{2^i \mid i \in \mathbb{Z}\}$. Thompson's groups T and V have analogous definitions. The group T is a collection of homeomorphisms of the circle, and V can be viewed as a group of homeomorphisms of the Cantor set. A good introduction to all of these groups is [2].

Recently, Ross Geoghegan posed the problem of determining whether the group F is Kähler (see [1]). A finitely presented group is called a *Kähler group* if it is the fundamental group of a compact Kähler manifold. The most important examples of Kähler groups (and perhaps the only ones) are the fundamental groups of smooth complex projective varieties.

Napier and Ramachandran soon produced proofs that F, T, and V are not Kähler groups [12]. Their proof in [13] that F is not Kähler used the fact that F is a strictly ascending HNN extension, and that such groups are never Kähler. Their proofs in [12] that T and V are not Kähler had two components. First, they showed that T and V have infinitely many filtered ends relative to certain subgroups (see [11] for the definition of filtered ends). Second, they appealed to the main theorem from [12], which implies that a Kähler group G having at least 3 filtered ends relative to some subgroup must have a quotient that is isomorphic to a hyperbolic surface group. Since the groups T and V are both simple [7], it is therefore clear that they cannot be Kähler.

The first half of their argument brought together a variety of sources, and used the theory of diagram groups over semigroup presentations [6], CAT(0) cubical complexes [4], and work of Thomas Klein on filtered ends of pairs of groups [10].

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The main purpose of this note is to simplify the first half of the argument, and strengthen the conclusion in the process. We prove the following theorem (definitions appear in Section 2):

Theorem 1.1. *The pairs* $(F, F_{[0,1/2]})$, $(T, T_{[0,1/2]})$, *and* $(V, V_{[0,1/2)})$ *all have infinitely many ends, where* $G_S = \{g \in G \mid g \text{ is the identity on } S\}$, *for* $G \in \{F, T, V\}$ *and* $S = [0, 1/2]$ *or* $[0, 1/2)$.

In Section 2, we let $e(G, H)$ denote the number of ends of the pair (G, H) (or the number of ends of G relative to H). Let $\tilde{e}(G, H)$ denote the number of filtered ends of the pair (G, H). The inequality $\tilde{e}(G, H) \geq e(G, H)$ holds true for any group G and subgroup H. The inequality can be strict [11], and it is in this sense that Theorem 1.1 strengthens the conclusions of [12]. Geoghegan [5] gives examples of pairs for which $e(G, H) = 3$ but $\tilde{e}(G, H) = \infty$. Note that we won't need to define the filtered ends of a group pair here.

Proposition 2.7 was originally proved in [9] (analogous results about filtered ends were proved in [11]). We include our own proof of this Proposition for the sake of completeness. As a result, the proof of Theorem 1.1 given here is largely self-contained, except for the main result of [3].

A second purpose of this note is to show that Thompson's groups T and V both have Serre's property FA, i.e., if T or V acts on a simplicial tree by automorphisms, then the action has a fixed point. The proof of this fact in Section 4 is due to Ken Brown. As a consequence, we answer a question posed by Mohan Ramachandran, who asked whether (or to what extent) the following conditions are equivalent for a finitely presented group G: (A) G has a finite index subgroup admitting a fixed-point-free action on a simplicial tree, and (B) the pair (G, H) has two or more ends, for some subgroup H. This question appears on the problem list maintained by Mladen Bestvina. Our results show that property (A) fails for T and V, although T and V have multiple (indeed, an infinite number) of ends relative to certain subgroups, and thus satisfy (B).

I would like to thank Mohan Ramachandran for encouraging me to publish a proof of Theorem 1.1. The combinatorial approach to group ends taken in Section 2 is indebted to [5]; I thank Ross Geoghegan for giving me a manuscript version of his book. After reading an earlier version of this paper, Mohan Ramachandran told me that Ken Brown had proved that T and V have property FA, and suggested the relevance of this fact to the above question. I thank Ken Brown for his notes (dating from the 1980s), which were the source of the argument in Section 4. Portions of this paper were written while I was visiting the Max Planck Institute for Mathematics. I thank the Institute for its hospitality and for the excellent working conditions during my stay.

2. Generalities About Ends of Graphs

Let Γ be a locally finite graph, i.e., a locally finite 1-dimensional CW complex. If $C \subseteq \Gamma$ is compact, then let $\mathrm{Comp}_{\infty}(\Gamma - C)$ denote the set of *unbounded* components of $\Gamma - C$, i.e., the components having non-compact closure. The *number of ends of* Γ, denoted $e(\Gamma)$, is

$$\sup_{C} \{|\mathrm{Comp}_{\infty}(\Gamma - C)|\}.$$

If G is a finitely generated group and S is a finite generating set, then $\Gamma_S(G)$, the *Cayley graph of G with respect to S*, is the graph having the group G as its

vertex set, and an edge $e(g, s)$ connecting g to gs for each $g \in G$ and $s \in S$. The *coset graph of $H\backslash G$ with respect to S*, denoted $\Gamma_S(H\backslash G)$, is the quotient of $\Gamma_S(G)$ by the natural left action of H.

If G is a finitely generated group, then the *number of ends of G*, denoted $e(G)$, is the number of ends of its Cayley graph $\Gamma_S(G)$, where S is some finite generating set. This definition doesn't depend on the choice of finite generating set, so we will often simply leave off the subscript S, and say that $e(G)$ is the number of ends of $\Gamma(G)$. In a similar way, we define the *number of ends of the pair* (G, H), denoted $e(G, H)$, by the equation $e(G, H) = e(\Gamma(H\backslash G))$.

2.1. **A generalization of Hopf's Theorem.** In [8], Heinz Hopf showed that an infinite group has 1, 2, or infinitely many ends. In this subsection we prove a generalization of this theorem. For this it will be helpful to have the following lemma.

Lemma 2.1. *If K is a compact subset of the locally finite graph Γ, then there is some finite connected subcomplex K' of Γ such that*

(1) $K \subseteq K'$;
(2) $|\mathrm{Comp}_\infty(\Gamma - K')| \geq |\mathrm{Comp}_\infty(\Gamma - K)|$, and
(3) each connected component of $\Gamma - K'$ is unbounded.

Proof. Suppose that K is a compact subset of Γ. Let K_1 be the the smallest subcomplex of Γ containing K. It follows from compactness of K that K_1 is a finite subgraph of Γ. We enlarge K_1 by adding a finite number of arcs to make the resulting graph, K_2, connected. Next, we add all connected components C of $\Gamma - K_2$ having compact closure to K_2. By the local finiteness of Γ and finiteness of K_2, the new subgraph K_3 is also compact, and now each component of $\Gamma - K_3$ is unbounded.

We set $K_3 = K'$. It is clear that (1) and (3) are satisfied; we need to check (2). Since $K \subseteq K'$, each connected component of $\Gamma - K'$ is contained in a (necessarily unique) connected component of $\Gamma - K$. If $|\mathrm{Comp}_\infty(\Gamma - K)| > |\mathrm{Comp}_\infty(\Gamma - K')|$ then there must be a connected component C of $\Gamma - K$ having non-compact closure and containing no such connected component C' of $\Gamma - K'$. Consider $C - K'$. The closure $\overline{C - K'}$ is a non-compact, locally finite graph. It follows that $C - K'$ contains a connected component C' having non-compact closure. Now C' is a connected component of $\Gamma - K'$ and $C' \subseteq C$; this is a contradiction. □

Lemma 2.2. *Let Γ be a locally finite graph. Let K_1, K_2 be disjoint finite connected subgraphs such that $\Gamma - K_1$ has m connected components $C_1, \ldots, C_m$ and $\Gamma - K_2$ has n connected components $C'_1, \ldots, C'_n$. If $K_1 \subseteq C'_1$ and $K_2 \subseteq C_1$, then $C_2, \ldots, C_m, C'_2, \ldots, C'_n$ are distinct connected components of $\Gamma - (K_1 \cup K_2)$.*

Proof. We first show that $C_2, C_3, \ldots, C_m, C'_2, C'_3, \ldots, C'_n$ are in fact components of $\Gamma - (K_1 \cup K_2)$. Choose a component C_i of $\Gamma - K_1$ ($2 \leq i \leq m$). Pick an arbitrary point $x \in C_i$; let $\widehat{C}$ be the component of $\Gamma - (K_1 \cup K_2)$ containing x. Pick an arbitrary point $y \in C_i$. Let $p_{xy} \subseteq C_i$ be the image of a path connecting x to y. Since $p_{xy} \subseteq C_i$, $p_{xy} \cap K_1 = \emptyset$ (because $C_i \subseteq \Gamma - K_1$) and $p_{xy} \cap K_2 = \emptyset$ (because $K_2 \subseteq C_1$ and $C_i \cap C_1 = \emptyset$). It follows that $C_i \subseteq \widehat{C}$. To prove the reverse inclusion, let $y \in \widehat{C}$ be arbitrary. Let $p_{xy} \subseteq \widehat{C}$ be the image of a path connecting x to y. It follows that $p_{xy} \subseteq \Gamma - (K_1 \cup K_2) \subseteq \Gamma - K_1$. This directly implies that

$y \in C_i$, so $\widehat{C} = C_i$. It follows (by symmetry in the case of $C'_2, \ldots, C'_n$) that each of $C_2, \ldots, C_m, C'_2, \ldots, C'_n$ is a connected component of $\Gamma - (K_1 \cup K_2)$.

It is easy to see that $C_2, \ldots, C_m$ are distinct components of $\Gamma - (K_1 \cup K_2)$, and that $C'_2, \ldots, C'_n$ are also distinct components of $\Gamma - (K_1 \cup K_2)$. Suppose $C_i = C'_j$ (for some $2 \leq i \leq m$, and $2 \leq j \leq n$). Let $x \in C_i$ and let $y \in C_{i'}$, where $i' \neq i$ and $2 \leq i' \leq m$. By the connectedness of K_1, there is a path whose image p_{xy} satisfies $p_{xy} \subseteq K_1 \cup C_i \cup C_{i'}$. It follows that $p_{xy} \subseteq \Gamma - K_2$, so in fact $p_{xy} \subseteq C'_j$. But now it follows that $p_{xy} \subseteq \Gamma - K_1$ (since $C'_j \cap K_1 \subseteq C'_j \cap C'_1 = \emptyset$), a contradiction. $\square$

Theorem 2.3. *If Γ is a locally finite graph admitting an infinite group of covering transformations, then $e(\Gamma) = 1, 2,$ or ∞.*

Proof. Suppose that $3 \leq e(\Gamma) < \infty$. It follows that there is some compact set $K \subseteq \Gamma$ such that $|\mathrm{Comp}_\infty(\Gamma - K)| = e(\Gamma) = n$. By the previous Lemma 2.1, we may assume that K is a finite connected subcomplex of K, and that each component of $\Gamma - K$ is unbounded.

Since Γ admits an infinite group of covering transformations, there must exist some covering transformation γ such that $\gamma \cdot K \cap K = \emptyset$. Let $C_1, \ldots, C_n$ denote the connected components of $\Gamma - K$; let $C'_1, \ldots, C'_n$ denote the connected components of $\Gamma - (\gamma \cdot K)$. All of these connected components are unbounded by our assumptions. By the connectedness of K, we can assume, without loss of generality, that $K \subseteq C'_1$. Similarly $\gamma \cdot K \subseteq C_1$, without loss of generality. It follows that $C_2, \ldots, C_n, C'_2, \ldots, C'_n$ are $2n-2$ unbounded components of $\Gamma - (K \cup \gamma \cdot K)$. Since $3 \leq e(\Gamma) = n$ and $2n - 2 \leq e(\Gamma)$, we have a contradiction. $\square$

Corollary 2.4. *If G is a finitely generated group, then $e(G) = 0, 1, 2,$ or ∞. If $H \leq G$ has infinite index in its normalizer $N_G(H) = \{g \in G \mid gHg^{-1} = H\}$, then $e(G, H) = 1, 2,$ or ∞.*

Proof. The first statement follows easily after applying the previous theorem to the Cayley graph $\Gamma(G)$. (The case $e(G) = 0$ corresponds to the case in which G is finite.) The second statement follows from applying Theorem 2.3 to the coset graph $\Gamma(H\backslash G)$ and noticing that $N_G(H)/H$ acts as covering transformations on $\Gamma(H\backslash G)$. $\square$

2.2. **The set of ends of a graph.** In certain situations, it is useful to work with a set of ends, rather than simply a number of ends. Let $c : [0, \infty) \to \Gamma$ be a cellular proper ray, i.e., each open interval $(i, i+1)$ (for $i \in \mathbb{Z}$) is mapped homeomorphically to an open edge by c, and $c^{-1}(K')$ is compact if K' is. Two cellular proper rays c and c' are joined by a *proper ladder* if there is a map

$$L : ([0, \infty) \times \{0\}) \cup ([0, \infty) \times \{1\}) \cup (Z \times [0, 1]) \to \Gamma$$

where Z is an infinite subset of the positive integers, $L(t, 0) = c(t)$, $L(t, 1) = c'(t)$, and $L^{-1}(K')$ is compact if K' is. We say that c and c' *define the same end*, and write $c \sim c'$, if c and c' are joined by a proper ladder. It is rather clear that $\sim$ is an equivalence relation; the equivalence classes are called *ends*. The set of ends of Γ is denoted $\mathcal{E}(\Gamma)$.

Proposition 2.5. *Let c_1, c_2 be proper rays in the locally finite graph Γ. The proper rays c_1, c_2 define the same end if and only if for any compact subset K of Γ there is some $t \in \mathbb{R}$ so that $c_1([t, \infty))$ and $c_2([t, \infty))$ are in the same component of $\Gamma - K$.*

Proof. ($\Rightarrow$) Suppose that c_1 and c_2 define the same end, and let K be a compact subset of Γ. Let ℓ be a proper ladder joining c_1 and c_2. Choose t_1 large enough that $c_1([t_1,\infty)) \cap K = c_2([t_1,\infty)) \cap K = \emptyset$. The properness of ℓ implies that some rung $\ell(\{t\} \times [0,1])$ of the ladder for $t > t_1$ is disjoint from K, and this directly implies that $c_1([t,\infty))$ and $c_2([t,\infty))$ are in the same component of $\Gamma - K$.

($\Leftarrow$) We need to connect c_1 to c_2 by a proper ladder. Begin by connecting $c_1(1)$ to $c_2(1)$ by an arbitrary arc, to form the first rung $\ell(\{1\} \times [0,1])$ of the ladder. Now we can use the hypothesis with $K = c_1([0,1]) \cup c_2([0,1]) \cup \ell(\{1\} \times [0,1])$ to add another rung to the ladder, which is disjoint from the first rung. By continuing in the same way, we inductively define a proper ladder between c_1 and c_2. □

Corollary 2.6. *If Γ is a locally finite graph and $e(\Gamma) = m > 0$, then there is some compact subset K of Γ such that $\Gamma - K$ has exactly m connected components $C_1, \ldots, C_m$, all of which are unbounded. Let K be any such compact subset. Two proper rays c_1, c_2 represent the same end if and only if $c_1([t,\infty)), c_2([t,\infty)) \subseteq C_i$ for some i and sufficiently large t.*

In particular, if $e(\Gamma)$ is finite, then $e(\Gamma) = |\mathcal{E}(\Gamma)|$.

Proof. The existence of K is an immediate consequence of the statement that $e(\Gamma) = m$ and Lemma 2.1. The forward direction of the second statement is clear.

Suppose c_1 and c_2 are two proper rays and $c_1([t,\infty)), c_2([t,\infty)) \subseteq C_i$ for some i and some t. If c_1 and c_2 define separate ends, then by Proposition 2.5 there is some compact $K' \subseteq \Gamma$ so that, for some t', $c_1([t',\infty))$ and $c_2([t',\infty))$ are contained in distinct components of $\Gamma - K'$. It follows from this that C_i contains two unbounded components C', C'' of $\Gamma - K'$. This implies that $|Comp_\infty(\Gamma - K')| > |Comp_\infty(\Gamma - K)| = e(\Gamma)$, a contradiction. □

2.3. **The case of two ends.**

Proposition 2.7. *Let Γ be a locally finite graph admitting an infinite group $C(\Gamma)$ of covering transformations. If $e(\Gamma) = 2$, then $C(\Gamma)$ has an infinite cyclic subgroup of finite index.*

Proof. Suppose $e(\Gamma) = 2$. Let K be a finite connected subgraph of Γ such that $\Gamma - K = C_1 \cup C_2$, where each C_i is an unbounded connected component of $\Gamma - K$. The group $C(\Gamma)$ acts on the set of ends, and after passing to a subgroup of index 2 if necessary, we can assume that $C(\Gamma)$ fixes both ends. Since Γ is infinite, locally finite, and $C(\Gamma)$ acts freely, there is some $\gamma \in C(\Gamma)$ so that $(\gamma \cdot K) \cap K = \emptyset$. Assume without loss of generality that $(\gamma \cdot K) \subseteq C_1$.

We first show that $(\gamma \cdot C_1) \cap C_2 = \emptyset$. Suppose $x \in (\gamma \cdot C_1) \cap C_2$; let $y \in (\gamma \cdot C_2) \cap C_2$ (Here $(\gamma \cdot C_2) \cap C_2 \neq \emptyset$ since γ fixes the ends of Γ). Since x and y are in C_2, there is an edge-path p connecting x to y in C_2, and $p \cap (\gamma \cdot K) \subseteq p \cap C_1 = \emptyset$. Thus $(\gamma^{-1} \cdot p)$ is an edge-path p connecting $(\gamma^{-1} \cdot x) \in C_1$ to $(\gamma^{-1} \cdot y) \in C_2$ and missing K. This is a contradiction, so $(\gamma \cdot C_1) \cap C_2 = \emptyset$.

Now note that $K \subseteq \gamma \cdot C_2$. For otherwise $K \subseteq \gamma \cdot C_1$, and since $\gamma \cdot C_1$ would then be an open set containing K, it would follow that $\gamma \cdot C_1$ contains elements C_2, a contradiction.

We can now apply Lemma 2.2: since $(\gamma \cdot K) \subseteq C_1$ and $K \subseteq (\gamma \cdot C_2)$, C_2 and $(\gamma \cdot C_1)$ are distinct connected components of $\Gamma - (K \cup (\gamma \cdot K))$, and both are clearly unbounded.

We have

$$\begin{aligned}\Gamma - (K \cup (\gamma \cdot K)) &= (C_1 \cap (\gamma \cdot C_1)) \cup (C_2 \cap (\gamma \cdot C_1)) \cup (C_1 \cap (\gamma \cdot C_2)) \\ &\quad \cup (C_2 \cap (\gamma \cdot C_2)) \\ &= C_2 \cup (\gamma \cdot C_1) \cup (C_1 \cap (\gamma \cdot C_2)).\end{aligned}$$

Indeed, the first equality above is obvious. The forward inclusion of the second equality follows from the fact that $(\gamma \cdot C_1) \cap C_2 = \emptyset$. The reverse inclusion follows from Lemma 2.2: since $\gamma \cdot K \subseteq C_1$, $K \subseteq (\gamma \cdot C_2)$, and K is compact and connected, C_2 and $(\gamma \cdot C_1)$ are distinct connected components of $\Gamma - (K \cup \gamma \cdot K)$ by the Lemma. Thus $C_2 \cup (\gamma \cdot C_1) \subseteq \Gamma - (K \cup \gamma \cdot K)$, and the reverse inclusion is established.

The second equality above easily implies that $C_2 \subseteq \gamma \cdot C_2$ and $\gamma \cdot C_1 \subseteq C_1$. Indeed, both of these last inclusions are proper: the first is proper since $K \subseteq \gamma \cdot C_2$, and the second is proper since $\gamma \cdot K \subseteq C_1$. This directly implies that γ has infinite order. Note also that $\overline{(C_1 \cap (\gamma \cdot C_2))}$ is compact, since $e(\Gamma) = 2$ and C_2, $\gamma \cdot C_1$ are both unbounded.

Next we need to show that, for any $x \in C_1$, there is $n < 0$ such that $\gamma^n \cdot x \in C_2$, and for any $x \in C_2$, there is $n > 0$ such that $\gamma^n \cdot x \in C_1$. We argue by contradiction: suppose $x \in C_1$ and $\{\gamma^{-1} \cdot x, \ldots, \gamma^{-n} \cdot x, \ldots\} \subseteq K \cup C_1$. Let $y \in C_2$; choose some path p connecting x to y. Now $\{\gamma^{-1} \cdot y, \ldots, \gamma^{-n} \cdot y, \ldots\} \subseteq C_2$, so each path $\gamma^{-1} \cdot p, \ldots, \gamma^{-n} \cdot p, \ldots$ meets K. It follows that some subsequence of $\gamma^{-1} \cdot x, \ldots, \gamma^{-n} \cdot x, \ldots$ has a limit (by the local finiteness of Γ), and thus infinitely many terms of the sequence are identical (since γ is a covering transformation), which implies that $\gamma^k = 1$ for some $k \neq 0$. This is a contradiction. Thus, for any $x \in C_1$, there is $n < 0$ so that $\gamma^n \cdot x \in C_2$. A similar argument shows that, for any $x \in C_2$, there is $n > 0$ so that $\gamma^n \cdot x \in C_1$.

Finally, we argue that $\widehat{K} = K \cup (\gamma \cdot K) \cup (C_1 \cap (\gamma \cdot C_2))$ is a compact fundamental domain for the action of $\langle \gamma \rangle$ on Γ; a standard argument then shows that $C(\Gamma)$ contains $\langle \gamma \rangle$ as a finite-index subgroup. Compactness of $\widehat{K}$ has already been established. Let $x \in \Gamma - K$. We may assume that $(\langle \gamma \rangle \cdot x) \cap K = \emptyset$. The argument of the previous paragraph shows that there is some $n_1 < 0$ so that $\gamma^{n_1} \cdot x \in C_2$ and some $n_2 > 0$ so that $\gamma^{n_2} \cdot x \in C_1$. Thus, there are consecutive integers k, $k+1$ such that $\gamma^k \cdot x \in C_2$ and $\gamma^{k+1} \cdot x \in C_1$. It follows that $\gamma^{k+1} \cdot x \in C_1 \cap (\gamma \cdot C_2)$. □

2.4. **Almost Invariant Subsets.** In the main argument of this paper, we will need a criterion, due to Sageev, for the pair (G, H) to have multiple ends. If G acts on a set S, then a subset T of S is said to be *almost invariant* if the symmetric difference $|gT \triangle T|$ is finite for any $g \in G$.

Theorem 2.8. [14] *Let G be a finitely generated group; let $H \leq G$. Consider the left (or right) action of G on the set G/H of left (or right) cosets of H. If there is a subset A of G/H such that*

(1) A is almost invariant, and
(2) each of A, A^c is infinite,

then $e(G, H) \geq 2$. Conversely, given a pair (G, H) such that $e(G, H) \geq 2$, there exists such an almost invariant set A in G/H.

Proof. ($\Rightarrow$) We prove the theorem in the case of the right action of G on the collection of right cosets $H\backslash G$. Begin by choosing a finite generating set S for G,

and building the coset graph $\Gamma_S(H\backslash G)$. Suppose there is a set A as in the statement of the theorem.

The elements of A can naturally be identified with vertices of $\Gamma_S(H\backslash G)$. Consider the collection of all edges e which connect an element of A with an element in A^c. We claim that the union K of all such edges is a finite subgraph of $\Gamma_S(H\backslash G)$. If not, then, by the finiteness of S and without loss of generality, there must be infinitely many disjoint directed edges of K, each labelled by the same generator $s \in S$, and each running from an element of A^c to an element of A. It follows from this that each of the (infinitely many) terminal vertices of such edges are in A, but not in As. This implies that $A \triangle As$ is an infinite set, which contradicts the fact that A is almost invariant.

Since $A - K$ and $A^c - K$ are both infinite sets, and they are clearly separated by K, it follows that $\Gamma_S(G, H)$ has at least two ends.

($\Leftarrow$) We won't need to use this implication, so we leave the (easy) proof as an exercise. The idea is to choose a compact subgraph K which divides Γ into at least two unbounded components, and then use the vertices of one of these components as A. $\square$

3. A proof that Thompson's groups have infinitely many relative ends

Lemma 3.1. *Let G_1 and G_2 be finitely generated groups. Suppose that $G_1 \leq G_2$, $H_1 \leq G_1$, $H_1 \leq H_2$, and $H_2 \leq G_2$.*

If the natural map $\phi : G_1/H_1 \to G_2/H_2$ is injective and $A \subseteq G_2/H_2$ is an almost invariant subset (under the left action of G_2), then $\phi^{-1}(A)$ is almost invariant under the left action of G_1.

Proof. Let A be an almost invariant subset of G_2/H_2 under the left action of G_2. We consider the inverse image $\phi^{-1}(A)$; let $g \in G_1$. We have that

$$\phi(\phi^{-1}(A) \triangle g\phi^{-1}(A)) \subseteq A \triangle gA.$$

Since ϕ is injective, it directly follows that $\phi^{-1}(A) \triangle g\phi^{-1}(A)$ is finite. $\square$

Proposition 3.2. *Let $V_{[0,1/2)}$ denote the subgroup of Thompson's group V which acts as the identity on $[0, 1/2)$.*

1. *The set $A = \{gV_{[0,1/2)} \mid g_{|[0,1/2)} \text{ is } affine\}$ is almost invariant under the action of V on $V/V_{[0,1/2)}$. Both A and its complement are infinite.*
2. *The quotient group $N\left(V_{[0,1/2)}\right)/V_{[0,1/2)}$ has no cyclic subgroup of finite index.*

In particular, $e(V, V_{[0,1/2)}) = \infty$.

Proof. (1) The statement that A is almost invariant is essentially the content of [3]. The main argument of [3] shows that $(v-1) \cdot \chi_A$ (where $\chi : P(V/V_{[0,1/2)}) \to \mathbb{Z}$ is the characteristic function) is a finite sum for any element $v \in V$. This clearly means that A is almost invariant.

For suitable selections of elements g_i (i a positive integer), g_i is affine on $[0, 1/2)$ and $g_i \cdot [0, 1/2)$ is the dyadic interval $[0, 2^{-i})$. For instance, we can let $g_i = x_0^i$,

where x_0 is one of the standard generators of $F \subseteq V$. As a piecewise linear homeomorphism of $[0,1]$, x_0 is defined as follows:

$$x_0(t) = \begin{cases} \frac{1}{2}t & 0 \leq t \leq 1/2 \\ t - \frac{1}{4} & 1/2 \leq t \leq 3/4 \\ 2t-1 & 3/4 \leq t \leq 1 \end{cases}$$

The cosets $g_i V_{[0,1/2)}$ are easily seen to be distinct, so A is infinite.

Any two distinct elements of the infinite subgroup $V_{[1/2,1)}$ represent distinct left cosets of $V_{[0,1/2)}$, and only one of these left cosets (containing the identity) lies in A. It follows that A^c is infinite. This proves (1).

(2) Each element of $V_{[1/2,1)}$ normalizes $V_{[0,1/2)}$, and any two elements in $V_{[1/2,1)}$ represent distinct left cosets of $V_{[0,1/2)}$. It follows that the group $V_{[1/2,1)}$ embeds in the quotient from the statement of the proposition. But $V_{[1/2,1)}$ is isomorphic to V itself, and V has no cyclic subgroup of finite index. This proves (2).

The final statement now follows from (1), (2), Theorem 2.8, Corollary 2.4, and Proposition 2.7. □

Proposition 3.3. $e(T, T_{[0,1/2]}) = e(F, F_{[0,1/2]}) = \infty$.

Proof. We argue that $e(F, F_{[0,1/2]}) = \infty$. The case of the group T is similar.

We first check the hypotheses of Lemma 3.1. It is clear that there are inclusions $F \to V$, $F_{[0,1/2]} \to V_{[0,1/2)}$. We next have to show that the induced map

$$\phi : F/F_{[0,1/2]} \to V/V_{[0,1/2)}$$

is injective. Suppose that $g_1 F_{[0,1/2]}$ and $g_2 F_{[0,1/2]}$ both have the same image under ϕ. It follows that $g_2 g_1^{-1} \in V_{[0,1/2)}$. Now clearly $g_2 g_1^{-1} \in F$, and it then follows from continuity that $g_2 g_1^{-1} \in F_{[0,1/2]}$, so $g_1 F_{[0,1/2]} = g_2 F_{[0,1/2]}$. Therefore ϕ is injective. This implies that $\phi^{-1}(A) = \{gF_{[0,1/2]} \mid g_{|_{[0,1/2]}} \textit{ is linear}\}$ is an almost invariant set.

We can prove that $\phi^{-1}(A)$ and $\phi^{-1}(A)^c$ are both infinite sets as in the proof of Proposition 3.2. Indeed, as before, the cosets $x_0^i F_{[0,1/2]}$ are all distinct (proving that $\phi^{-1}(A)$ is infinite) and any two distinct elements of $F_{[1/2,1]}$ define distinct cosets of $F_{[0,1/2]}$, and exactly one of these cosets ($F_{[0,1/2]}$ itself) is in $\phi^{-1}(A)$. This proves that $\phi^{-1}(A)^c$ is also an infinite set. It follows that $e(F, F_{[0,1/2]}) = 2$ or ∞.

As in Proposition 3.2, there is an embedding of $F_{[1/2,1]}$ into the quotient group

$$N(F_{[0,1/2]})/F_{[0,1/2]},$$

and $F_{[1/2,1]} \cong F$. Since F has no infinite cyclic subgroup of finite index, it follows that $e(F, F_{[0,1/2]}) = \infty$. □

4. Proof that T and V have Serre's property FA

Suppose that G acts simplicially on the simplicial tree Γ. We say that G acts *without inversions* if, whenever $g \in G$ leaves an edge e invariant, g acts as the identity on e. We will assume (after barycentrically subdividing the tree, if necessary) that any simplicial action of G on a tree is an action without inversions. We say that G has *property FA* if every simplicial action of G on a tree has a fixed point, i.e., $G \cdot v = v$, for some $v \in \Gamma$.

We will need some standard facts about automorphisms of trees. If $g \in G$, then $Fix(g) = \{x \in \Gamma \mid g \cdot x = x\}$. The set $Fix(g)$ is a subtree of Γ if it is non-empty. If $Fix(g) \neq \emptyset$, then g is called *elliptic*; otherwise, g is *hyperbolic*.

Lemma 4.1. *Let G be a group acting on a simplicial tree Γ by automorphisms.*

(1) Let $g \in G$. Either g acts on a unique simplicial line in Γ by translation (called an axis *for g), or $Fix(g) \neq \emptyset$.*
(2) If the fixed sets $Fix(g_1), Fix(g_2)$ are non-empty and disjoint, then $Fix(g_1g_2) = \emptyset$.
(3) If g_1 and g_2 are elliptic and $g_1 \cdot Fix(g_2) = Fix(g_2)$, then $Fix(g_1) \cap Fix(g_2) \neq \emptyset$.
(4) If G is generated by a finite set of elements $s_1, \ldots, s_m$ such that the s_j and the s_is_j have fixed points, then G has a fixed point.

Proof. (1) is a consequence of Proposition 24 (page 63) from [15]. Our proof of (2) uses Corollary 1 from page 64 of [15], which says that if $abc = 1$ and each of a, b, c is elliptic, then a, b, and c have a common fixed point. If we read this corollary with $g_1 = a$, $g_2 = b$, and $g_2^{-1}g_1^{-1} = c$, then the assumption that g_1g_2 is elliptic leads to a contradiction, since $Fix(g_1) \cap Fix(g_2) = \emptyset$. To prove (3), we use Lemma 9 from page 61 of [15], which asserts that, if Γ_1 and Γ_2 are disjoint subtrees of Γ, then there is a unique minimal geodesic segment ℓ connecting Γ_1 to Γ_2. That is, if ℓ' connects a vertex of Γ_1 with a vertex in Γ_2, then $\ell \subseteq \ell'$. It is fairly clear that ℓ meets each of the subtrees Γ_1 and Γ_2 in exactly one point. Now suppose that g_1 and g_2 are elliptic; we assume that their fixed sets are disjoint. Let $\ell = [x, y]$ connect $x \in Fix(g_1)$ with $y \in Fix(g_2)$. We assume that ℓ is the minimal geodesic connecting these fixed sets. Our assumptions imply that $[x, y] \cup g_1 \cdot [x, y]$ is a geodesic segment connecting two points in $Fix(g_2)$, namely y and $g_1 \cdot y$. Since $Fix(g_2)$ is a tree, $[x, y] \cup g_1 \cdot [x, y] \subseteq Fix(g_2)$. This implies the contradiction $Fix(g_1) \cap Fix(g_2) \neq \emptyset$, proving (3).

Statement (4) is Corollary 2 from [15], page 64. □

Throughout the rest of this section, we assume that $G = T$ or V. Let $g \in G$. We say that g is *small* if g is the identity on some *standard dyadic subinterval of* $[0, 1]$ (i.e., some subinterval of the form $[\frac{i}{2^n}, \frac{i+1}{2^n}]$, where n is a non-negative integer and $0 \leq i < 2^n$).

Lemma 4.2. *If G acts on a tree Γ, and $g \in G$ is small, then g has a fixed point.*

Proof. Let $g \in G$. Suppose, for a contradiction, that g is hyperbolic and acts by translation on the geodesic line ℓ. Let I be a standard dyadic subinterval of $[0, 1]$ such that $g_{|I} = id_I$. We consider the subgroup H of G having support in I. (This group is isomorphic either to F or to V, depending on whether G is T or V, respectively.) For any $h \in H$, $hgh^{-1} = g$, so

$$g \cdot h\ell = hgh^{-1} \cdot h\ell = h \cdot g\ell = h \cdot \ell.$$

It follows from the uniqueness of the axis ℓ that $h\ell = \ell$. Thus, the entire group H leaves the line ℓ invariant, so there is a homomorphism $\phi : H \to D_\infty$. The kernel of ϕ is large: it will contain $[H, H]$ (if $G = T$, in which case H is isomorphic to F, and every proper quotient of F is abelian by Theorem 4.3 from [2]) or H itself (if $G = V$, in which case H is isomorphic to V, and therefore simple). By Theorem 4.1 from [2], $[H, H]$ consists of the subgroup of H which acts as the identity in neighborhoods of the left- and right-endpoints of I. Thus, if I' is a standard dyadic interval whose closure is contained in the interior I, then any element of G supported in I' will fix the entire line ℓ.

We can choose a conjugate kgk^{-1}, where $k \in G$, and the support of kgk^{-1} is contained in I'. It follows that kgk^{-1} is elliptic, and therefore g is elliptic as well. This is a contradiction. □

Theorem 4.3. *T has Serre's property FA.*

Proof. Identify $[0,1]/\sim$ with the standard unit circle $S^1 = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 = 1\}$ by the quotient map $f : [0,1] \to S^1$, where $f(t) = (\cos(2\pi t), \sin(2\pi t))$. This identification induces an action of T on S^1. We consider four subgroups of T:

$$\begin{aligned} T_L &= \{g \in T \mid g \cdot (x,y) = (x,y) \ if \ x \geq 0\} \\ T_R &= \{g \in T \mid g \cdot (x,y) = (x,y) \ if \ x \leq 0\} \\ T_U &= \{g \in T \mid g \cdot (x,y) = (x,y) \ if \ y \leq 0\} \\ T_D &= \{g \in T \mid g \cdot (x,y) = (x,y) \ if \ y \geq 0\} \end{aligned}$$

Each of these groups is isomorphic to F, and so can be generated by two elements, which are necessarily small as elements of T. Moreover $T = \langle T_L, T_R, T_U, T_D\rangle$. It follows that T is generated by 8 elements, each of which is small, such that the product of any two generators is also small. This implies that T fixes a point, by Lemma 4.1, (4). □

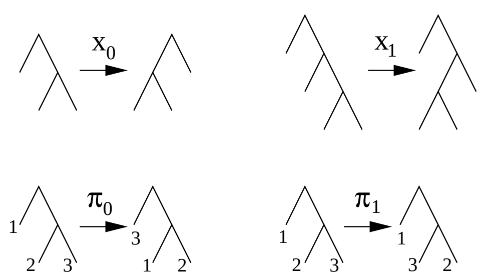

FIGURE 1. A generating set for Thompson's group V. The elements x_0, x_1, and π_0 generate Thompson's group T.

Theorem 4.4. *The group V has Serre's property FA.*

Proof. We recall from [2] that $T = \langle x_0, x_1, \pi_0\rangle$ and $V = \langle T, \pi_1\rangle$, where x_0, x_1, π_0, and π_1 appear in Figure 1.

Let V act on a tree Γ. By the previous theorem, we know that T has a fixed point. Thus, the elements x_0, x_1, and π_0 all have fixed points, and any two of these elements will have a common fixed point. The element π_1 has finite order, so it must be elliptic. It will be sufficient (by Lemma 4.1 (4)) to show that each product $\pi_1\pi_0$, $\pi_1 x_0$, and $\pi_1 x_1$ is elliptic.

First, we note that $\pi_1\pi_0$ is an element of finite order, so it must be elliptic. Next, we note that $\pi_1 x_1$ is small, so it must be elliptic.

It is routine to check that $x_0^{-1}\pi_1 x_0 \pi_1$ is small, and therefore elliptic. It follows that $Fix(x_0^{-1}\pi_1 x_0) \cap Fix(\pi_1) \neq \emptyset$. Now

$$\begin{aligned} Fix(x_0^{-1}\pi_1 x_0) \cap Fix(\pi_1) \neq \emptyset &\Rightarrow x_0^{-1}(Fix(\pi_1)) \cap Fix(\pi_1) \neq \emptyset \\ &\Rightarrow Fix(x_0^{-1}) \cap Fix(\pi_1) \neq \emptyset. \end{aligned}$$

This implies that $\pi_1 x_0$ is elliptic. □

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